Notes On The Klein-Gordon Equation

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Abstract

In this article, we derive the scalar Klein-Gordon equation from the formal information theory framework. The least biased probability distribution is obtained, and the scalar equation is recast in terms of a Fokker-Planck equation in terms of the imaginary time, or a Schroedinger equation for the proper time. This method yields the Green's function parametrized by the proper-time, and is related to the results of Schwinger and Dewitt. The derivation can then allow the use of potentials as constraints along with the Hamiltonian or moments of the evolution. The information the- oretic, analogously the maximum entropy method, also allows one to ex- amine the possibility of utilizing generalized and non-extensive statistics in the derivation. This approach yields non-linear evolution in the Klein- Gordon-like partial differential equations. Furthermore, we examine the Klein-Gordon equation in curved space-time, and we compare our results to results obtained from path integral approaches.

1 Introduction

In this article, we derive the scalar Klein-Gordon equation from within the information theory (or maximum entropy) framework. The least biased probability distribution is obtained from the maximization of the entropy with suitable con- straints. The probability then evolves under the scalar wave equation with a parametrized mass term, the Klein-Gordon equation parametrized by the proper time. The scalar equation is recast as a Fokker-Planck equation in terms of the parametrization of imaginary time, or a Schroedinger equation for the proper time. This method is akin to the Feynman parametrization of the mass term to solve the Klein-Gordon wave equation and the derivation of the photon Green's function, as well as the Schwinger-Dewitt proper-time formalism. The derivation also allows one to use a higher dimensional version of the methods of quantum mechanics, specifically, the use of potentials in a linear superposition with the Hamiltonian of the evolution via the use of observable constraints in the entropy maximization procedure. We then use the method to examine the underlying stochastic differential equations and the microscopic evolution. We explore the solution to the wave equation for a particle moving

in curved spacetime and given the moments obtained from the underlying stochastic microsocpic evolution, and obtain the short-time transitional (conditional) probability from the maximum entropy variational principle.

In the past, the proper-time formalism has gained acceptance in the treatment of the relativistic wave equation both in flat and curved spacetime. The method simply involves a parametrization of the mass term, and recasts the Klein-Gordon equation into a partial differential equation (PDE) with an extra dimension (here) being that of the parametrization (the proper time). The solution is then obtained by transforming from the proper-time to the mass term [2, 3]. Also, well known analytic continuation methods allow us to recast the result in terms of a higher dimension Schroedinger equation with the subsequent connection to known methods of low energy quantum mechanics and the Schwinger-Dewitt proper time formalism. What has not been done is to consider the resultant partial differential equation as a diffusion-like equation in the context of probability theory and the maximum entropy method [5, 7], or equivalently from within information theory [1]. A mathematical reason for this is that the equations of the wave equation and Klein-Gordon form of equations are elliptical equations, and maximum entropy and information theory derive PDEs that are parabolic, of the diffusion, Fokker-Planck and Schroedinger forms, and that have 'friction'-like terms that allow for equilibriation with a heat bath in the thermodynamics perspective.

This interpretation of the relativistic elliptical wave equation and Klein-Gordon equations as parametrized higher dimensional parabolic equations following the work of Feynman will allow us to recast the problem as one of maximum entropy and information theory. From derived least biased distributions and their evolution parabolic PDEs we will obtain an equivalent description of the problem in terms of the underlying stochastic differential equations. For the curved-spacetime case this will mean two new formulations, an extensive statistics and the recently developed nonextensive statistics formulation of the stochastic gravity approach. The moments obtained from the stochastic description can then be used to obtain the short-time transition probability for the evolution. The Klein-Gordon partial differential equation is

$$\frac{m^2c^4}{\hbar^2}\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial(-it)^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}, \tag{1}$$

and we rewrite the imaginary time as $-it = \tau$. We can obtain this equation from the maximum entropy method, or equivalently the information theoretic framework as the information measure is equivalent to the entropy with appropriate constants of proportionality relating the bits of information to the measure of disorder the entropy measured by the Boltzmann constant. We begin the derivation by stating that we know the moments of the variables (x, y, z, τ) and maximize the Gibbs-Boltzmann entropy $< S >= -\int PlnP$, equivalently minimizing the Shannon information measure, given the following constraints parametrized by λ

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x,y,z,\tau,\lambda) dxdydzd\tau = D_x \lambda \tag{2}$$

$$\langle y^2 \rangle = \int_{-\infty}^{\infty} y^2 P(x,y,z,\tau,\lambda) dxdydzd\tau = D_y \lambda \tag{3}$$

$$\langle z^2 \rangle = \int_{-\infty}^{\infty} z^2 P(x,y,z,\tau,\lambda) dxdydzd\tau = D_z \lambda \tag{4}$$

$$\langle \tau^2 \rangle = \int_{-\infty}^{\infty} \tau^2 P(x,y,z,\tau,\lambda) dxdydzd\tau = D_\tau \lambda. \tag{5}$$

Maximizing the entropy we obtain the least biased probability distribution function (PDF) normalized with N

$$P(x,y,z,\tau,\lambda) = N e^{-\frac{x^2}{2D_x\lambda}} e^{-\frac{y^2}{2D_y\lambda}} e^{-\frac{z^2}{2D_z\lambda}} e^{-\frac{\tau^2}{2D_\tau\lambda}}. \tag{6}$$

This PDF solves the diffusion-like partial differential equation PDE

$$\frac{\partial P(x,y,z,\tau,\lambda)}{\partial \lambda} = \frac{D_\tau}{2}\frac{\partial^2 P}{\partial \tau^2} + \frac{D_x}{2}\frac{\partial^2 P}{\partial x^2} + \frac{D_y}{2}\frac{\partial^2 P}{\partial y^2} + \frac{D_z}{2}\frac{\partial^2 P}{\partial z^2}. \tag{7}$$

We can rewrite the PDF via a Laplace or alternatively Fourier transform as

$$P(\vec{x},\tau,\lambda) = \int e^{-\varphi\lambda} P(\vec{x},\tau,\varphi) d\varphi \tag{8}$$

and substitute in the PDE Eq. (1) and obtain

$$-\varphi P(x,y,z,\tau,\varphi) = (\frac{D_\tau}{2}\frac{\partial^2}{\partial \tau^2} + \frac{D_x}{2}\frac{\partial^2}{\partial x^2} + \frac{D_y}{2}\frac{\partial^2}{\partial y^2} + \frac{D_z}{2}\frac{\partial^2}{\partial z^2}) P(x,y,z,\tau,\varphi). \tag{9}$$

With the identification of the diffusion constants as $D_{x,y,z} = 2$ and $D_\tau = 2c^{-2}$ and $\phi = -\frac{m^2c^4}{\hbar^2}$, we recover the Klein-Gordon equation as in Eq.(1). We can also obtain a 5D Schroedinger equation using this method. The PDE is recast in this form if we take the time-like parameter and Wick rotate about the imaginary axis to $\lambda = -i\alpha$. The resulting PDE is

Schroedinger-like

$$i\frac{\partial P(x,y,z,\tau,\alpha)}{\partial\alpha}=\frac{D_\tau}{2}\frac{\partial^2P}{\partial\tau^2}+\frac{D_x}{2}\frac{\partial^2P}{\partial x^2}+\frac{D_y}{2}\frac{\partial^2P}{\partial y^2}+\frac{D_z}{2}\frac{\partial^2P}{\partial z^2}, \tag{10}$$

and we can identify the diffusion constants as relating the 'mass' to the variance as $D_{x,y,z}=-\frac{\hbar}{m^*}=\sigma^2$ and $D_\tau=-\frac{\hbar}{c^2m^*}$. Alternatively, one can work with the Hamiltonian directly. From the Legendre transform of the relativistic Lagrangian, one can obtain the generalized canonical momenta, and write a relativistic Hamiltonian and any potentials as the observables to be used as constraints in the entropy maxmization procedure. Thus the constraints will be the $< H >$, and any potentials to be included such as external potentials or interactions $< V >$, and the entropy $< S >$ is maximized to yield the least biased probability. In the case of $< V >= 0$ and flat space-time, the least biased probability will solve the massive Klein-Gordon equation as in Eq.(1).

2 Connection with nonextensive statistics

Having obtained a maximum entropy derivation of the KG type equation, a question arises as to the form of the equations obtained if one were to use a generalized entropy. Instead of the Gibbs-Boltzmann entropy which yields the extensive statistics, one can use a generalized form of the entropy, the nonextensive entropy, which has become well known after the work of C. Tsallis [5, 7] and which yields non-linear partial differential equations for the evolution of the distribution. In cosmology, the non-extensive statistics have been shown to model 'small' systems such as galactic matter distributions, where the range of the interaction (re: gravitational) is on the same scale as the system size [10], the solar neutrino problem [9] and systems that exhibit complex and non-linear dynamics [11]. The present derived result should prove of interest as a new nonlinear theory as nonlinear Klein-Gordon models are of importance in high energy particle physics. The non-extensive statistical entropy, or analogously the incomplete information theoretic measure [1] is

$$\langle S\rangle_q=-\frac{1-\int P^q(\vec{x},\tau,\lambda)d\vec{x}d\tau}{1-q}. \tag{11}$$

This form of entropy is known to yield power law distributions. Maximizing the entropy with the observable moments as constraints yields the least biased power law distribution,

$$P(\vec{x},\tau,\lambda)=\frac{1}{Z(\lambda)}[1+\beta(\lambda)(q-1)(\vec{x}^2+\tau^2)]^{\frac{-1}{q-1}}, \tag{12}$$

where $Z^2(\lambda)\beta(\lambda) = const.$ and $Z(\lambda)$ is the partition function and is related to the normalization as usual. This distribution solves the nonlinear partial differential equation

$$\frac{\partial P(\vec{x},\tau,\lambda)}{\partial \lambda} = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \tau^2})P(\vec{x},\tau,\lambda)^{2-q}, \tag{13}$$

where we have set $\hbar = c = 1$ and absorbed any constants into the variables (x, y, z, τ). We can obtain the solution of the Klein-Gordon-like equation if we Laplace or alternatively Fourier transform according to

$$P(\vec{x},\tau,\phi) = \int e^{-\varphi\lambda} P(\vec{x},\tau,\lambda) d\lambda. \tag{14}$$

Some comments should be made regarding Eq.(13). The nonlinear form of the PDE Eq.(13) is one that by q-parametrization is a controllable deviation from the Gaussian solution of $q- > 1$ towards the power law solution of Eq.(12). The nonlinear terms of the traditional nonlinear Klein-Gordon equations of particle high energy physics are solved by functions that are deviations from the Gaussian, squeezed or stretched Gaussians. These nonlinear deviations are present by parametrized nonlinearity q and the PDE equation Eq.(13) should be considered as a new class of nonlinear Klein-Gordon equations. Another comment is to be made regarding curved space-time applications of this equation Eq.(13). The curvature terms are included as the diffusion coefficients of the PDE Eq.(4) as the curvature tensor terms are for diagonal tensors or tensors that can be transformed to diagonal tensors the inverses of the diffusion coefficients. These curvature terms cause the PDE Eq.(4) to become nonlinear and the solutions of which become deviations from the Gaussian as discussed. Therefore the possibility exists here that the complicated nonlinearities due to nonlinear terms included to describe high energy particles and nonlinear terms introduced to describe curved space-time can be described or modeled by the q-parametrized nonlinearity of Eq.(13) which has known solutions of the power-law form of Eq.(12). Also it is important to note that a recent advance in nonextensive quantum mechanics [12] is promising for a relativistic Schroedinger-like PDE approach to nonextensive relativistic quantum mechanics. This will be pursued in subsequent work.

3 Stochastics

The diffusion equation Eq.(4) is a Fokker-Planck equation. This macroscopic evolution equation for the PDF allows us to obtain the underlying Ito-Langevin stochastic differential equations [4]. The individual trajectories are

$$\begin{aligned} dx &= \sqrt{D_x} dW(\lambda)_x, \qquad & dy &= \sqrt{D_y} dW(\lambda)_y \\ dz &= \sqrt{D_z} dW(\lambda)_z, \qquad & d\tau &= \sqrt{D_\tau} dW(\lambda)_\tau, \end{aligned} \tag{15}$$

where the $dW_{x,y,z,\tau}$ are Wiener processes, and are composed of Gaussian white noise with delta correlations. The case of the non-extensive statistics derived Klein-Gordon-like PDE Eq.(13) is somewhat more complicated. The underlying stochastic evolution is of the statistical feedback form [6], and from Eq. (13) we obtain

$$
\begin{aligned}
dx &= \sqrt{D_x P^{1-q}(\vec{x},\tau,\lambda)}dW(\lambda)_x, \qquad dy = \sqrt{D_y P^{1-q}(\vec{x},\tau,\lambda)}dW(\lambda)_y \\
dz &= \sqrt{D_z P^{1-q}(\vec{x},\tau,\lambda)}dW(\lambda)_z, \qquad d\tau = \sqrt{D_\tau P^{1-q}(\vec{x},\tau,\lambda)}dW(\lambda)_\tau.
\end{aligned}
\tag{16}
$$

We note again that the macroscopic distribution PDF appears in the underlying microscopic stochastic evolution. This statistical feedback 'loop' between the macroscopic evolution of probability distribution *and* the microscopic stochastic trajectory(s) can be viewed as the source of the nonlinearity in the PDE from this perspective. The modification of the trajectories via coupling to the large scale macroscopic evolution, alternatively the memory effect or the historical trends squeezes the PDF from the usual Gaussian type into a power-law, and can be a prototype for nonlinear deviations from the usual parametrized Klein-Gordon based Gaussian statistics. We note again that due to the presence of curvature, metrics with nonlinear terms will contribute terms to the Klein-Gordon equation that will cause deviations from a Gaussian [2] and that can be modeled by the q-parametrized nonlinearity. We will examine the regular form of this evolution in the context of curved spacetime in the next section. Mapping the regular curved space-time parametrized evolution to the nonextensive statistics q-parametrized evolution we leave for subsequent work.

4 Klein-Gordon Equation in Curved Spacetime

The Klein-Gordon equation in curved spacetime from within the proper-time formalism of Schwinger and Dewitt has been treated, at least from the Gaussian approximation, by Bekenstein [2, 3]. The equation is derivable from an action principle

$$
I = \frac{1}{2}\int d^3x dt \{\sqrt{-g}(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - 2\alpha R\phi^2)\}
\tag{17}
$$

The Euler-Lagrange equation then yields the Klein-Gordon equation in curved space-time, with $c = \hbar = 1$

$$
(g^{\mu\nu}\partial_\mu\partial_\nu - m^2 - \xi R)\phi = 0.
\tag{18}
$$

We will solve the general case first, with $R = R(\vec{x})$. The solution method will depend on the use of stochastic calculus and the maximum entropy method. We define the two-point function $P(\vec{x},\tau,\lambda;\vec{x}',\tau',\lambda')$.

This function evolves in Eq. (18). Writing generally, the backwards PDE is

$$\frac{\partial P}{\partial \lambda} = \left(b_{\nabla}^2 \nabla^2 + b_{\tau}^2 \frac{\partial^2}{\partial \tau^2} - \xi R \right) P(\vec{x}, \tau, \lambda | \vec{x}', \tau', \lambda'). \tag{19}$$

We can transform this equation into a standard form if we transform according to

$$\alpha(\vec{x}, \tau) = \int^{x} \frac{1}{\sqrt{b_x^2(\vec{x}', \tau)}} dx', \qquad \beta(\vec{x}, \tau) = \int^{y} \frac{1}{\sqrt{b_y^2(\vec{x}', \tau)}} dy',$$
$$\gamma(\vec{x}, \tau) = \int^{z} \frac{1}{\sqrt{b_z^2(\vec{x}', \tau)}} dz'. \tag{20}$$

The transformed equation is

$$\frac{\partial P}{\partial \lambda} = \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \gamma^2} \right) P + b_{\tau}^2 \frac{\partial^2 P}{\partial \tau^2} - \xi R P. \tag{21}$$

We redefine the function P as $P(\vec{\alpha}, \tau, \lambda) = \pi e^{-U(\vec{\alpha})}$. Upon substitution and gathering terms, we have

$$\begin{aligned} \frac{\partial \pi}{\partial \lambda} =& \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \gamma^2} \right) \pi + b_{\tau}^2 \frac{\partial^2 \pi}{\partial \tau^2} + (-2) \frac{\partial U}{\partial \alpha} \frac{\partial \pi}{\partial \alpha} \\ &+ (-2) \frac{\partial U}{\partial \beta} \frac{\partial \pi}{\partial \beta} + (-2) \frac{\partial U}{\partial \gamma} \frac{\partial \pi}{\partial \gamma} + \frac{\partial^2 U}{\partial \alpha^2} \pi + \frac{\partial^2 U}{\partial \beta^2} \pi + \frac{\partial^2 U}{\partial \gamma^2} \pi - \xi R \pi. \end{aligned} \tag{22}$$

The curvature term is eliminated if we make the identification of the first term with ξR. Upon replacing the original arguments, we have

$$\begin{aligned} \frac{\partial \pi}{\partial \lambda} =& \left(b_x^2 \frac{\partial^2}{\partial x^2} + b_y^2 \frac{\partial^2}{\partial y^2} + b_z^2 \frac{\partial^2}{\partial z^2} + b_{\tau}^2 \frac{\partial^2 P}{\partial \tau^2} \right) \pi + (-2) b_x^4 \frac{\partial U}{\partial x} \frac{\partial \pi}{\partial x} \\ &+ (-2) b_y^4 \frac{\partial U}{\partial y} \frac{\partial \pi}{\partial y} + (-2) b_z^4 \frac{\partial U}{\partial z} \frac{\partial \pi}{\partial z}. \end{aligned} \tag{23}$$

This equation is now in the backwards Chapman-Kolmogorov form. The two-point function, or the conditional distribution, is a solution of this equation, as it is a solution of the forward Chapman-Kolmogorov equation, the Fokker-Planck equation,

$$\begin{aligned} \frac{\partial}{\partial \lambda} \pi(\vec{x}, \tau, \lambda | \vec{x}', \tau', \lambda') =& \left(\frac{\partial^2}{\partial x^2} b_x^2 + \frac{\partial^2}{\partial y^2} b_y^2 + \frac{\partial^2}{\partial z^2} b_z^2 + \frac{\partial^2}{\partial \tau^2} b_{\tau}^2 \right) \pi - \frac{\partial}{\partial x} [F_x(\vec{x}) \pi] \\ &- \frac{\partial}{\partial y} [F_y(\vec{x}) \pi] - \frac{\partial}{\partial z} [F_z(\vec{x}) \pi] \qquad , \end{aligned} \tag{24}$$

where we have renamed the drift coefficients as $F_x(\vec{x}) = 2b^4{}_x \frac{\partial U}{\partial x}$, and similarly for the other coefficients and have dropped the prime superscripts of the variables. The underlying stochastic differential equations are

$$\begin{aligned} dx &= F_x(\vec{x})d\lambda + \sqrt{b_x^2(\vec{x}',\tau)}dW_x(\lambda), \qquad & dy &= F_y(\vec{x})d\lambda + \sqrt{b_y^2(\vec{x}',\tau)}dW_y(\lambda), \\ dx &= F_z(\vec{x})d\lambda + \sqrt{b_z^2(\vec{x}',\tau)}dW_z(\lambda), \qquad & d\tau &= \sqrt{b_\tau^2(\vec{x}',\tau)}dW_\tau(\lambda), \end{aligned} \tag{25}$$

where the dW are Wiener processes and are delta correlated as before. The time averages of the descretized stochastic processes are ($\Delta x = x - x_o$)

$$\begin{aligned} \langle \Delta \mathrm{x} \rangle &= F_x(\vec{x}')\Delta\lambda, \qquad & \left\langle \Delta x^2 \right\rangle &= b_x^2(\vec{x}',\tau')\Delta\lambda, \\ \langle \Delta y \rangle &= F_y(\vec{x}')\Delta\lambda, \qquad & \left\langle \Delta y^2 \right\rangle &= b_y^2(\vec{x}',\tau')\Delta\lambda, \\ \langle \Delta z \rangle &= F_z(\vec{x}')\Delta\lambda, \qquad & \left\langle \Delta z^2 \right\rangle &= b_z^2(\vec{x}',\tau')\Delta\lambda, \\ \langle \Delta \tau \rangle &= 0, \qquad & \left\langle \Delta \tau^2 \right\rangle &= b_\tau^2(\vec{x}',\tau')\Delta\lambda. \end{aligned} \tag{26}$$

These averages coincide with conditional expectation values such as

$$\begin{aligned} \langle \Delta x | \vec{x}', \tau', \lambda \rangle &= \int \Delta x \pi(\Delta\vec{x}, \Delta\tau, \Delta\lambda | \vec{x}', \tau', \lambda') d\Delta\vec{x} d\Delta\tau = F_x(\vec{x}')\Delta\lambda, \\ \langle \Delta x^2 | \vec{x}', \tau', \lambda \rangle &= \int \Delta x^2 \pi(\Delta\vec{x}, \Delta\tau, \Delta\lambda | \vec{x}', \tau', \lambda') d\Delta\vec{x} d\Delta\tau = b_x^2(\vec{x}', \tau')\Delta\lambda, \end{aligned} \tag{27}$$

and so on for the other coordinate conditional expectation values. We solve for the short-time conditional (transition) probability distribution as follows. The conditional variances about the means are now the constraints as we maximize the conditional entropy

$$S(\vec{x}', \tau', \lambda') = -\int \pi(\Delta\vec{x}, \Delta\tau, \Delta\lambda | \vec{x}', \tau', \lambda') ln \pi(\Delta\vec{x}, \Delta\tau, \Delta\lambda | \vec{x}', \tau', \lambda') d\Delta\vec{x} d\Delta\tau. \tag{28}$$

Explicitly we maximize the following expression

$$\delta[\langle S \rangle] + \delta\left[\beta_x \left\langle (\Delta x - \langle \Delta x \rangle)^2 \right\rangle + \beta_y \left\langle (\Delta y - \langle \Delta y \rangle)^2 \right\rangle\right] + \left[\beta_z \left\langle (\Delta z - \langle \Delta z \rangle)^2 \right\rangle + \beta_\tau \left\langle \Delta \tau^2 \right\rangle\right] \tag{29}$$

The maximization yields the least biased probability

$$\pi(\Delta\vec{x}, \Delta\tau, \Delta\lambda | \vec{x}', \tau', \lambda') = \frac{1}{Z(\lambda)} e^{-\left(\beta_x (\Delta x - \langle \Delta x \rangle)^2 + \beta_y (\Delta y - \langle \Delta y \rangle)^2 + \beta_z (\Delta z - \langle \Delta z \rangle)^2 + \beta_\tau \Delta \tau^2\right)}. \tag{30}$$

The Lagrange multipliers are obtained from the partition function via functional relationships between the multipliers and the moments. As an example, the x-coordinate multipliers are

$$-\frac{\partial \ln Z}{\partial \beta_x} = \left\langle \Delta x^2 \right\rangle, \tag{31}$$

and the partition function is $Z = \int \pi$. Upon integration, this yields

$$Z = \sqrt{\frac{2\pi}{\beta_x}} \sqrt{\frac{2\pi}{\beta_y}} \sqrt{\frac{2\pi}{\beta_z}} \sqrt{\frac{2\pi}{\beta_\tau}}, \tag{32}$$

and we can solve for the multipliers using the relationships Eq.(31) to obtain

$$\beta_x = \frac{1}{2\left(\Delta x - \langle\Delta x\rangle\right)^2} = \frac{1}{2b_x(\vec{x}', \tau')^2\Delta\lambda}, \tag{33}$$

and similarly for the multipliers in the other coordinates, in terms of their moments. Gathering the terms together and substituting for the moments, we have

$$P(\Delta\vec{x}, \Delta\tau, \lambda|\vec{x}', \tau', \lambda') =$$
$$e^{-U(\vec{x})} \frac{e^{-\frac{(\Delta x - F_x(\vec{x}')\Delta\lambda)^2}{2b_x^2(\vec{x}',\tau')\Delta\lambda}}}{\sqrt{2\pi b_x^2(\vec{x}', \tau')\Delta\lambda}} \frac{e^{-\frac{(\Delta y - F_y(\vec{x}')\Delta\lambda)^2}{2b_y^2(\vec{x}',\tau')\Delta\lambda}}}{\sqrt{2\pi b_y^2(\vec{x}', \tau')\Delta\lambda}} \frac{e^{-\frac{(\Delta z - F_z(\vec{x}')\Delta\lambda)^2}{2b_z^2(\vec{x}',\tau')\Delta\lambda}}}{\sqrt{2\pi b_z^2(\vec{x}', \tau')\Delta\lambda}} \frac{e^{-\frac{(\Delta\tau)^2}{2b_\tau^2(\vec{x}',\tau')\Delta\lambda}}}{\sqrt{2\pi b_\tau^2(\vec{x}', \tau')\Delta\lambda}}, \tag{34}$$

We can compare this result with the proper-time formalism of Schwinger-Dewitt and the results of Bekenstein [2, 3] if we wick rotate to the proper time $\lambda = -is$. The Green's function from the path integral approach is

$$G(\Delta\vec{x}, \Delta\tau, \lambda) = i\,(4\pi is)^{-2}\, e^{i\frac{\sigma(\vec{x},\vec{x}')}{2s}} \Delta^{\frac{1}{2}}[1 + f_1 is + f_2(is)^2 + ...]. \tag{35}$$

The factors in the expression are $\sigma(\vec{x};\vec{x}') = \sigma(\vec{x}';\vec{x})$ and stands for half the proper distance squared $dS^2 = g_{\mu\nu}dx_\mu dx_\nu$, or minus half the proper time squared, and contains the metric coefficients, which we have re-written as inverses and are contained in the diffusion coefficients ${b^2}_{x,y,z,\tau}$. The quantity Δ is $\Delta(\vec{x};\vec{x}') = -g(\vec{x})^{\frac{1}{2}} D_{VM} g(\vec{x}')^{\frac{1}{2}}$, and D_{VM} is the VanVleck-Morreta determinant $D_{VM} = Det(-\frac{\partial^2\sigma}{\partial x^\mu \partial x^\nu})$. The flat space-time result is $G(\Delta\vec{x}, \Delta\tau, \lambda)_o = i(4\pi is)^{-2} e^{i\frac{\pi_O(\vec{x};\vec{x}')}{2\sigma}} \Delta^{\frac{1}{2}}$, and Bekenstein solves the Gaussian case for the geodesic extreme paths and obtains

$$G(\Delta\vec{x}, \Delta\tau, \lambda) = i\,(4\pi is)^{-2}\, e^{i\frac{\sigma(\vec{x},\vec{x}')}{2s}} e^{-i[\xi - \frac{1}{6}]Rs} \Delta(\vec{x}, \vec{x}')^{\frac{1}{2}}. \tag{36}$$

We note again that the diffusion coefficients are the inverse of the metric coefficients and are here included in the factor σ. The normalizations are expectedly different, as is the R curvature scalar term, as the solution in the present case takes into account terms higher in the expansion. The short-time transition probability is accurate to order $\theta(\Delta\lambda)$. We regain the flat space-time case where $R = 0$ if we let the means vanish (the drift coefficients), as the scalar curvature is included in the x, y, z, τ coordinates' means, and assume constant (unit) diffusion coefficients.

5 Conclusion

In this article we have derived the scalar Klein-Gordon equation from within themaximumentropymethod,obtaining the (Wick rotated) proper-time analog and solutions. We have also generalized the approach to include other than Gibbs-Boltzmann entropies, namely the non-extensive statistics entropy of C. Tsallis. This resulted in a new nonlinear form of the Klein-Gordon equation, with its nonlinear proper time PDE. We then examine the Klein-Gordon equa- tion in curved space-time, and obtain a general solution in which the diffusion coeffients are related to the inverse metric coefficients. Also, the curvature scalar is then included in the drift. We compare our result with the results of Bekenstein and Dewitt. We leave the application to specific curved space-time metrics to future work.

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